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TEMPERATURE AND HEAT FLUX DISTRIBUTIONS IN INCOMPRESSIBLE TURBULENT EQUILIBRIUM BOUNDARY LAYERS

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Abstract— The concepts of equilibrium boundary layers are used to determine the temperature distributions and heat flux in turbulent incompressible boundary layers. The effects of wall heat flux gradients, free stream pressure gradients and varying turbulent Prandtl number are considered and discussed. The two cases of heat transfer and adiabatic wall are analysed and the Reynolds Analogy Factor and Recovery Factor are computed. A comparison is made between the turbulent shear stress and the turbulent heat flux. Considerations are confined to the outer, fully developed turbulent part of the boundary layers.

NOMENCLATURE

A_1, A_2, A_3, D, variable coefficients defined by equations (26)–(28), (34) and (35);
B, wake parameter in equation (14);
C, empirical constant in skin friction equation (16);
c_f, local skin friction coefficient;
c_p, specific heat at constant pressure;
I, shape parameter, equation (13);
K, constant in asymptotic form of velocity defect, equation (17);
P, average pressure;
Pr, molecular Prandtl number;
q, heat flux (positive from the fluid to the wall);
r, recovery factor;
Re_x, Reynolds number based on downstream flow distance;
Re_{δ^*}, Reynolds number based on displacement thickness;
St, Stanton number;
t, hypothetical temperature defined by equations (4) and (5);
T, average temperature;
u, v, fluctuating components of velocity;
U, V, components of average velocity;
u_τ, friction velocity;
w, Coles' wake function, equation (12);
x, y, space coordinates (flow in x direction);
a, b, constant in temp. "law of the wall", equations (42) and (47);
δ, total boundary-layer thickness;
δ^*, boundary-layer displacement thickness;
Δ, Clauser thickness, equation (2);
κ, von Kármán constant in velocity, equation (11) ($\approx 0{\cdot}41$);
Π, pressure gradient parameter, $\Pi = (\Delta\omega/\tau_w)\, dP_\infty/dx$, equation (1);
ρ, density;
σ_t, turbulent Prandtl number;
τ, turbulent shear stress;
Φ, temperature defect function;
φ, dimensionless shear stress function, equation (20);
Ψ, variable in temperature defect parameter, equations (31) and (32);
Ω, wall heat flux gradient parameter, $\Omega = \Delta/\omega[(dq_w/dx)/q_w]$, equation (6a);
ω, $u_\tau/U_\infty = \sqrt{(c_f/2)}$;

G

η, y/Δ;
ξ, defined by equation (3).

Subscripts
a, adiabatic wall case;
q, heat-transfer case;
s, total or stagnation conditions;
w, conditions at wall;
∞, free stream conditions.

Superscripts
$'$, denotes differentiation with respect to η.

1. INTRODUCTION

AT THIS stage in the development of turbulent boundary-layer research, no single theory has yet evolved which can completely define all of the flow variables which exist in turbulent flows. Much of the most recent work in this field has involved the use of finite difference methods to solve the turbulent boundary-layer equations [1, 2]. However, even these methods must rely on a certain amount of empirical data and hypothetical relations to make the particular equations used a determinate set. In addition, the finite difference methods themselves can be considered experimental because, in many cases, these methods lack a rigorous mathematical proof that they converge to the true solutions of the boundary-layer equations. In spite of these drawbacks, the finite difference methods seem to be one of the most reasonable approaches provided that they are used in conjunction with valid physical concepts. One such physical concept is the basis for the present work and is commonly referred to as the concept of equilibrium boundary layers. This concept simply states that there are certain types of boundary layers in which the velocity profiles are similar in shape at various positions, x, and only differ by a scale factor in U and y [3]. This implies that parameters which relate to the shape of the velocity profiles do not vary with x. Such boundary layers can also exist in the presence of pressure gradients, as long as the pressure gradient parameter, Π, remains constant. The exact experimental results of Clauser [4] have verified the existence of such boundary layers and the theoretical development of this concept was expanded by Rotta [3].

The basic principles of equilibrium boundary layers are applied here to the problem of determining the temperature distributions in incompressible turbulent boundary layers. This represents an expansion of the earlier work of Rotta [11] and includes the additional effects of free stream pressure and wall heat flux gradients. In a more recent article [12], Alber and Coats considered the effects of free stream pressure and wall heat flux gradients. However, their investigation was limited to the case of heat transfer. The approach used by Alber and Coats was based on the application of an effective viscosity concept, which was hypothesized by Mellor and Gibson [15], to develop a family of equilibrium enthalpy profiles. In the present report, the mean velocity is given analytically and the turbulent heat flux is related to the turbulent shear stress by the concept of turbulent Prandtl number. The influence of a variable turbulent Prandtl number and a constant turbulent Prandtl number different from unity is considered. In addition, the problem is solved, not only for the heat transfer case, but also for the case of the adiabatic wall. (This case is sometimes referred to as the wall thermometer problem, since the wall temperature is assumed to be in equilibrium with the surrounding mediums, much the same as a thermometer.)

The development of the basic theory described in this report and the method of transforming the boundary layer equations were taken from the notes of Dr. J. C. Rotta, Aerodynamische Versuchsanstalt, Göttingen. The computations, analysis and discussion of the results were done by the author.

2. STATEMENT OF THE PROBLEM

In this work, considerations are confined to the fully developed turbulent part of the boundary layer and the viscous sublayer is assumed to be very small in comparison to the total boundary layer thickness. The problem is formulated by asking what the temperature distribution at any point, x, is, given that the velocity distribution and certain free stream and boundary conditions are known.

By keeping in mind that the equilibrium conditions of constant pressure gradient and shape parameter must be satisfied, a pressure gradient parameter, Π, is defined as

$$\Pi = \frac{\Delta\omega}{\tau_w}\frac{dP_\infty}{dx} = -\frac{\Delta}{\omega}\frac{dU_\infty/dx}{U_\infty} = \text{Constant}, \quad (1)$$

where Δ is the Clauser thickness defined by

$$\Delta = \int_0^\infty \frac{U_\infty - U}{u_\tau}\,dy. \quad (2)$$

In order to satisfy the equilibrium condition, $d\Delta/dx$ must be constant. In similar fashion, the concept of similarity can be applied to the temperature distribution and a dimensionless parameter, ξ, can be defined by

$$\xi = \frac{\Delta}{\omega}\frac{dt/dx}{t}. \quad (3)$$

The quantity t is a hypothetical temperature which is defined, for the adiabatic wall case, by

$$t_a = \frac{u_\tau U_\infty}{2c_p} \quad (4)$$

so that, in this case, $\xi = -2\Pi$. For the case of heat transfer, Squire [5] termed t a "friction temperature" in analogy to the friction velocity, u_τ, and he defined it by

$$t_q = \frac{-q_w}{\rho c_p u_\tau}. \quad (5)$$

Substitution of equation (5) into equation (3) yields

$$\xi = \frac{\Delta}{\omega}\frac{dq_w/dx}{q_w} + \Pi. \quad (6)$$

The first term in this equation defines the variation of the wall heat flux with x. For this reason, it will be defined as the wall heat flux gradient parameter,

$$\Omega = \frac{\Delta}{\omega}\frac{dq_w/dx}{q_w}, \quad (6a)$$

which is analogous to the pressure gradient parameter defined by equation (1).

These parameters are used, in combination with an assumed velocity profile, to transform the simplified boundary-layer equations into an ordinary differential equation with a temperature function as the independent variable.

3. DEVELOPMENT OF THE EQUATIONS

With the standard assumptions for incompressible turbulent boundary layers [6], the defining flow equations can be written as:

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (7)$$

Momentum

$$\rho\left[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right] = \frac{\partial \tau}{\partial y} - \frac{dP}{dx}, \quad (8)$$

Energy

$$c_p\rho\left[U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y}\right] = \frac{\partial q}{\partial y} + \tau\frac{\partial U}{\partial y} + U\frac{dP}{dx}. \quad (9)$$

3.1 *Velocity distribution*

Since only the region outside of the sublayer is being considered, the velocity defect law, defined by

$$F = \frac{U_\infty - U}{u_\tau} \quad (10)$$

can be applied. In this regard, a logarithmic velocity profile is assumed, so that

$$F = -\frac{1}{\kappa}\left[\ln\left(\frac{y}{\delta}\right) - B(2 - w)\right] \quad (11)$$

where w is Coles wake function, which can be approximated by

$$w = 39\left(\frac{y}{\delta}\right)^3 - 125\left(\frac{y}{\delta}\right)^4 + 183\left(\frac{y}{\delta}\right)^5 - 133\left(\frac{y}{\delta}\right)^6 + 38\left(\frac{y}{\delta}\right)^7. \quad (12)$$

The wake factor, B, is a free parameter and is a constant which can be determined from a characteristic shape parameter of the defect law. This characteristic parameter is defined as

$$I = \int_0^\infty F^2\, d\eta\,. \quad (13)$$

parameter, Π. Through experimental and theoretical considerations Nash [7] derived the following relationship:

$$I = 6{\cdot}1\sqrt{(\Pi + 1{\cdot}81)} - 1{\cdot}7\,. \quad (15)$$

As the sublayer is approached, it is usually assumed that the velocity profile conforms to the "law of the wall" and is described by the equation,

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right) + C\,, \left(\frac{yu_\tau}{\nu} > 50\right) \quad (16)$$

where C is a constant which depends on the

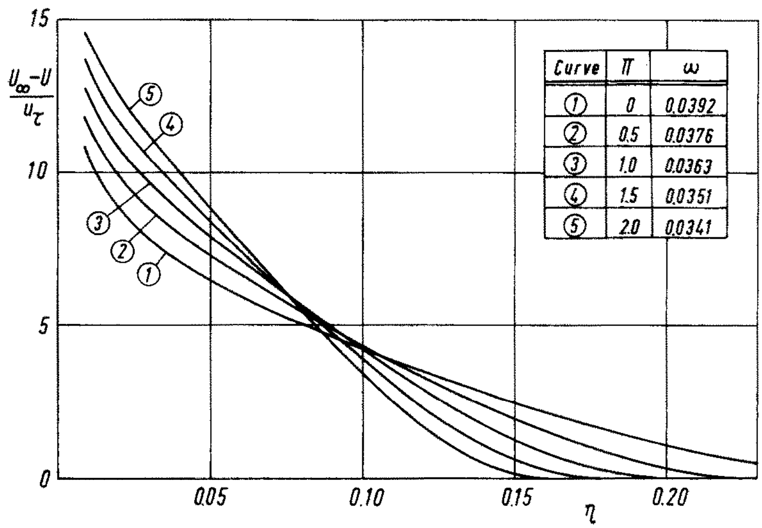

FIG. 1. Velocity defect profiles with adverse pressure gradients.

Through integration of the square of the velocity defect law, equation (11), over the boundary-layer thickness, the shape parameter is expressed as a function of B, namely

$$I = \frac{2 + 3{\cdot}2B + 1{\cdot}522B^2}{\kappa(1 + B)}. \quad (14)$$

For equilibrium boundary layers this parameter must be considered constant with x. Clauser [4] has postulated that the shape parameter, I, must be a function of the pressure gradient

surface roughness. There is a region where the inner profile equation (16) and the outer profile equation (11) approach each other asymptotically. Therefore, if these two equations are set equal to each other, a relationship for the skin friction coefficient can be obtained:

$$\frac{1}{\omega} = \sqrt{\left(\frac{1}{c_f/2}\right)} = \frac{1}{\kappa}\ln Re_\delta^* + C + K\,, \quad (17)$$

where K is determined from the wake parameter by

$$K = -\frac{1}{\kappa}\ln\left(\frac{1+B}{\kappa}\right) + \frac{2B}{\kappa},$$

and Re_δ^* is the Reynolds number based on the displacement thickness [8]. Equation (16) can also be expressed in terms of F and η by

$$\lim_{\eta \to 0} F = -\frac{1}{\kappa}\ln \eta + C + K. \qquad (18)$$

Some examples of the velocity defect profiles, computed from equation (11) are shown in Fig. 1. The values of ω shown in Fig. 1 were computed from equation (17) by assuming a Reynolds number, based on the displacement thickness, of 5500.

3.2 *Turbulent Prandtl number distribution*

The concept of the turbulent Prandtl number relates the exchange of momentum to the exchange of heat in turbulent flows. Specifically, it is the ratio of turbulent eddy viscosity to eddy heat conductivity. With the aid of this definition, the turbulent shear stress can be related to the turbulent heat flux by

$$\frac{\tau_t}{q_t} = \frac{\sigma_t}{c_p}\frac{\partial U/\partial y}{\partial T/\partial y}, \qquad (19)$$

where σ_t is the turbulent Prandtl number. Earlier theories have postulated that the turbulent Prandtl number remained nearly constant and was, in fact, equal to 1. The relatively exact measurements made by H. Ludwieg [9] in a circular pipe have refuted this postulation and have shown a decrease in σ_t from a value of about 0·9 at the wall to approximately 0·67 in the center of the pipe. This indicates that the turbulent momentum exchange increases more than the turbulent heat exchange as the level of turbulence increases. The results of *H.* Reichardt [10], from measurements in a free jet, indicated values of σ_t as low as 0·5. There is, however, no universal relationship which one can use to specify the turbulent Prandtl number at a given point. Therefore, the distributions assumed in this work are strictly arbitrary and the intent is merely to show the effect of a varying turbulent Prandtl number on the temperature distributions. The arbitrary distributions assumed in this work are listed in Table 1.

Table 1. Assumed variation of σ_t

1. constant	$\sigma_t = \sigma_{tw}$
2. convex	$\sigma_t = 0{\cdot}9 - 0{\cdot}4(y/\delta)^2$
3. linear	$\sigma_t = 0{\cdot}9 - 0{\cdot}4\,(y/\delta)$
4. linear decreasing	$\sigma_t = \sigma_{tw} - (\sigma_{tw} - 0{\cdot}5)(y/\delta)$

3.3 *Shear stress distribution*

The shear stress distribution is calculated by first defining a dimensionless shear stress parameter, φ,

$$\varphi = -\frac{\rho\overline{uv}}{\tau_w} = \frac{\tau}{\tau_w}. \qquad (20)$$

This quantity is then substituted in the momentum equation (8) and the velocity in the y direction is eliminated by use of the continuity equation (7). By applying the equilibrium boundary layer assumption and the velocity defect law, the following equation for φ' results.

$$\varphi' = \Pi[2F - \omega F^2] + \left(\frac{1}{\omega}\frac{d\Delta}{dx} - \Pi\right)\left(\eta - \omega\int_0^\eta F\, d\eta'\right)F', \qquad (21)$$

where $\eta = \Delta/y$ and $d\Delta/dx$ is a constant, which is determined by integrating the above equation over the entire boundary layer. The result of this integration is

$$\frac{d\Delta}{dx} = \frac{\omega}{1 - \omega I}[\Pi(3 - 2\omega I) + 1]. \qquad (22)$$

The shear stress at any point, η, can be found by integrating equation (21) from 0 to η:

$$\varphi = 1 + 2\Pi\int_0^\eta F\, d\eta' - \Pi\omega\int_0^\eta F^2\, d\eta' - \left(\frac{1}{\omega}\frac{d\Delta}{dx} - \Pi\right)$$

$$\times \left[\int_0^\eta F \, d\eta' - \omega \int_0^\eta F^2 \, d\eta' - F \left(\eta - \omega \int_0^\eta F \, d\eta' \right) \right]. \tag{23}$$

4. TRANSFORMATION OF BOUNDARY LAYER EQUATIONS

In a manner analogous to the velocity defect concept, a "temperature defect parameter", Φ, is defined as

$$\Phi = (T - T_\infty)/t, \tag{24}$$

where T is the average fluctuating temperature, and t was previously defined in equations (4) and (5). The energy equation (9) can be expressed in terms of non-dimensional quantities through the use of equations (11), (19), (20) and (24). After performing the necessary substitutions, the result is a second order, ordinary differential equation, linear in Φ:

$$\Phi'' + A_1\Phi' + A_2\Phi + A_3 = 0 \tag{25}$$

where the coefficients are defined by

$$A_1 = \frac{-F''}{F'} + \frac{\varphi'}{\varphi}(1 - \sigma_t) - \frac{\sigma_t'}{\sigma_t} + \frac{\Pi\sigma_t}{\varphi} \times (2F - \omega F^2), \tag{26}$$

$$A_2 = (1 - \omega F)\frac{\sigma_t \xi F'}{\varphi}, \tag{27}$$

and

$$A_3 = \frac{U_\infty^2 \omega^2 \sigma_t}{c_p t} F'^2. \tag{28}$$

If the boundary conditions are now specified, the problem will be mathematically formulated. Use is made of the definition of the turbulent Prandtl number to specify a boundary condition as the wall is approached. For the adiabatic wall case, this condition is

$$\lim_{y \to 0} \frac{\partial T}{\partial y} = -\sigma_{tw} \frac{U}{c_p} \frac{\partial U}{\partial y},$$

or

$$\lim_{\eta \to 0} \Phi' = 2\sigma_{tw}(1 - \omega F) F'. \tag{29}$$

Similarly, for the heat transfer case, the condition is

$$\lim_{y \to 0} \Phi' = \sigma_{tw} F'. \tag{30}$$

If these boundary conditions are integrated, the result for the adiabatic wall is

$$\Phi = \sigma_{tw}(2 - \omega F) F + \Psi_a \tag{31}$$

and the heat transfer

$$\Phi = \sigma_{tw} F + \Psi_q, \quad \text{where} \quad \lim_{\eta \to 0} \Psi = \text{constant}. \tag{32}$$

By generalizing these equations to apply over the entire boundary layer, the problem can be formulated in terms of Ψ by substituting equations (31) and (32) into equation (25), so that

$$\Psi'' + A_1\Psi' + A_2\Psi + D = 0, \tag{33}$$

where, for the adiabatic wall case,

$$D_a = 2\sigma_{tw}A_1(1 - \omega F) F' + \sigma_{tw}A_2(2 - \omega F) \times F + 2\sigma_{tw}F''(1 - \omega F) - 2\sigma F''(1 - \omega F) \tag{34}$$

and for heat transfer

$$D_q = \sigma_{tw}A_1F' + \sigma_{tw}A_2F + \sigma_{tw}F''. \tag{35}$$

The boundary condition as the wall is approached

$$\lim_{\eta \to 0} \Psi = \text{constant}. \tag{36}$$

The other boundary condition can be defined at the outer edge of the boundary layer as

$$\Psi = 0; \qquad \eta = \delta/\Delta. \tag{37}$$

It should be remarked here that one reason for formulating the problem in terms of Ψ, is that this function rather clearly illustrates the influence of Π, Ω and σ_t on the correspondence between the temperature and velocity gradients. Consideration of equations (25) and (33) and the coefficients A_1, A_2 and D reveals that, when $\sigma_t = 1{\cdot}0 = \text{constant}$ and $\Pi = \Omega = 0$, there is a one to one correspondence between the temperature and velocity defects, that is, $\Phi = F$. This is, of course, to be expected from the definition of the turbulent Prandtl number. However, if any of these three functions are varied, this condition no longer exists. The additional significance of these functions will become evident in the paragraphs that follow.

5. DISCUSSION OF RESULTS

Equation (33), with boundary conditions (36) and (37), was solved numerically using a backward difference method which was started at the outer edge of the boundary layer. In applying the finite difference method, two important aspects of the problem must be considered. First, there is a singularity at the outer edge of the boundary layer ($\eta = \delta/\Delta$). Second the asymptotic form of the velocity defect equation (18) is only valid outside of the sublayer ($yu_\tau/\nu > 50$). Therefore, the value of η must be restricted so that it does not become less than approximately $50/Re_\delta^*$. The finite difference method was used to calculate the values of Ψ, Ψ' and Ψ'' from $\eta = \delta/\Delta$ to $50/Re_\delta +$. The values of Ψ at the wall ($\eta = 0$) were then found by extrapolation. As the value of Π is increased, the value of η at the outer edge of the boundary layer decreases (see Fig. 1). This decreases the range of values which η can assume, if Re_δ^* is not increased. The size of the η increment, for a given value of Ω or Π, was determined by repeating the calculations for different values of the increment until the boundary condition $\Psi' = 0$ was most nearly approached as η decreased to the wall. All calculations in this report were made at an assumed Reynolds number, Re_δ^*, of 5500. Within the range of values of Π which were considered, this corresponds to local skin friction coefficients, c_f, between 0·0032 and 0·00226. Although the effects of ω on the results are not specifically illustrated in this report, the calculations showed that the assumed values of ω or Re_δ^* have a very small effect on the results. This fact was previously noted in the original report by Rotta [11]. In order to show the independent effects of a wall heat flux gradient and a free stream pressure gradient, the calculations were made by letting Π assume finite values while $\Omega = 0$, and vice versa. These effects are discussed independently in the paragraphs that follow.

5.1 *Effect of wall heat flux gradients*

The effects of a constant heat gradient parameter on the temperature distribution

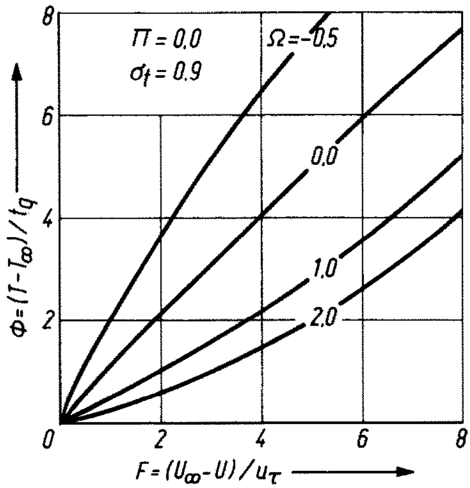

FIG. 2. Influence of the wall heat flux gradient parameter (Ω) on the temperature defect profiles with constant turbulent Prandtl number ($\sigma_t = 0{\cdot}9$) and zero pressure gradient ($\Pi = 0$).

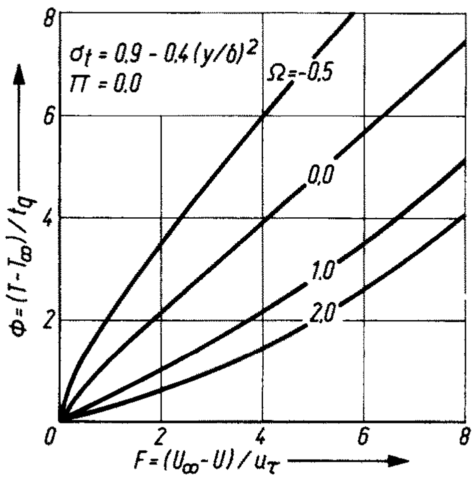

FIG. 3. Influence of the wall heat flux gradient parameter (Ω) and variable turbulent Prandtl number (σ_t) on the temperature defect profiles with zero pressure gradient ($\Pi = 0$).

through an incompressible equilibrium boundary layer are illustrated in Figs. 2–4. These curves were computed for a zero pressure gradient, $\Pi = 0$, so that $\xi = \Omega$, see equation (6). The case of constant σ_t is shown in Fig. 2, while Figs. 3 and 4 show the additional influence of a variable σ_t. In all cases, higher

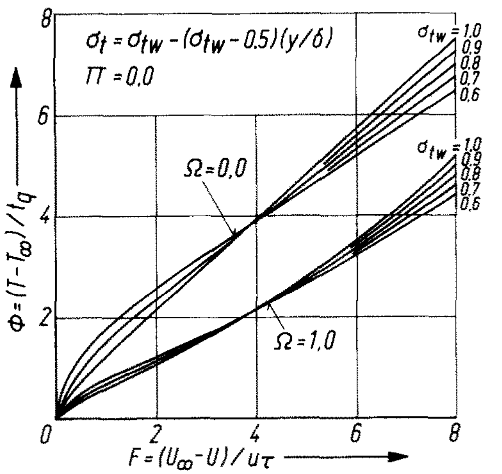

FIG. 4. Relative influence of the wall heat flux gradient parameter (Ω) and a variable turbulent Prandtl number (σ_t) on the temperature defect profiles with zero pressure gradient ($\Pi = 0$).

values of Ω (corresponding to the case of increasing wall heat flux with x), result in smaller temperature differences through the boundary layer. Variation of the turbulent Prandtl number has a relatively insignificant effect on the temperature defect profiles (compare Figs. 2 and 3). However, the assumed values or distributions of σ_t increase in importance as the wall is approached (Fig. 4). Further examples of these effects can be demonstrated through the use of the dimensionless heat flux ratio, q/q_w. This ratio can be derived by expressing equation (19) in terms of the non-dimensional quantities Φ, F, φ and σ_t. To accomplish this, equations (10), (20) and (24) are substituted into equation (19) with the results that

$$q = \frac{-\rho U_\infty \omega t c_p \varphi \Phi'}{\sigma_t F'}. \qquad (38)$$

Use is then made of equation (15) to eliminate t, so that

$$q/q_w = \frac{\varphi \Phi'}{\sigma_t F'}. \qquad (39)$$

Some examples of how this ratio varies, for different values of Ω and σ_t, are shown in Figs. 5 and 6 for $\Pi = 0$. Comparison of these

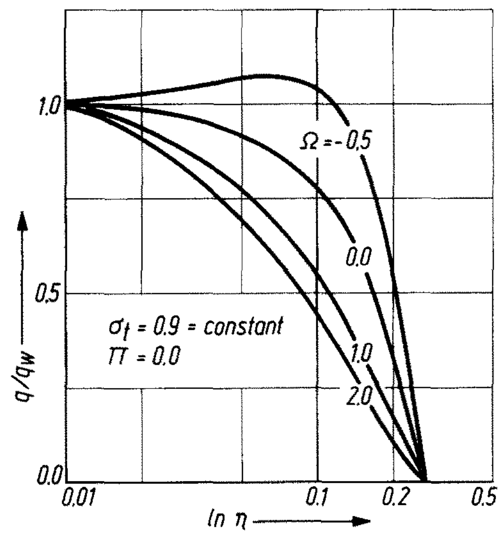

FIG. 5. Effect of the wall heat flux gradient parameter (Ω) on the heat flux distribution with constant turbulent Prandtl number ($\sigma_t = 0{\cdot}9$) and zero pressure gradient ($\Pi = 0$).

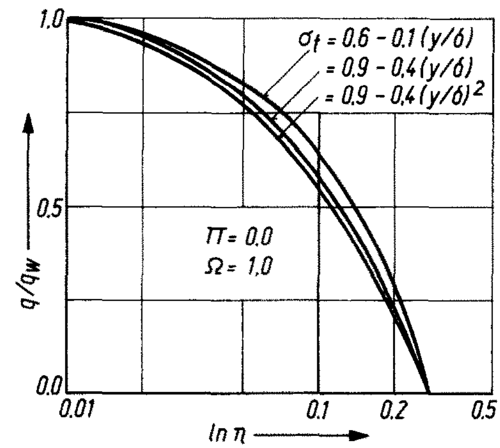

FIG. 6. Effect of a variable turbulent Prandtl number (σ_t) on the heat flux distribution with a wall heat flux gradient ($\Omega = 1$) and zero pressure gradient ($\Pi = 0$).

two figures rather clearly illustrates the small influence of σ_t and the significant influence of Ω on the heat flux in the boundary layer.

5.2 *Effect of free stream pressure gradient*

As the free stream pressure gradient parameter is increased, the temperature defect

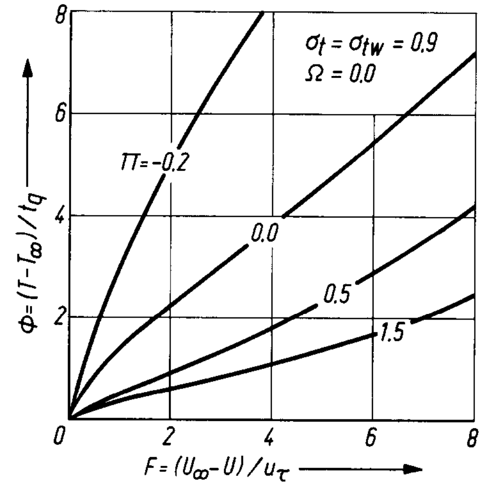

FIG. 7. Influence of the pressure gradient parameter (Π) on the temperature defect profile with constant turbulent Prandtl number (σ_t = 0·9) and zero wall heat flux gradient (Ω = 0).

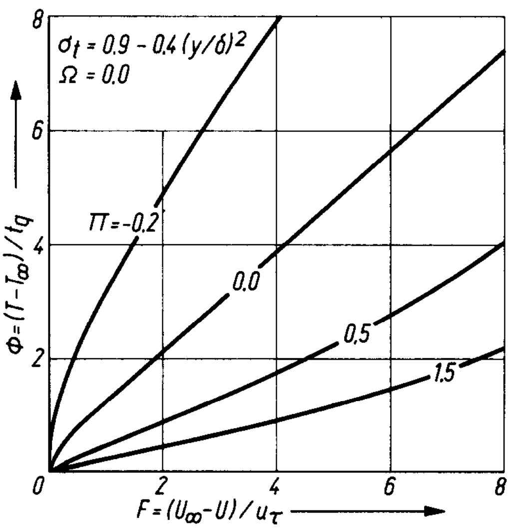

FIG. 8. Influence of the pressure gradient parameter (Π) and a variable turbulent Prandtl number (σ_t) on the temperature defect profiles with zero wall heat flux gradient (Ω = 0).

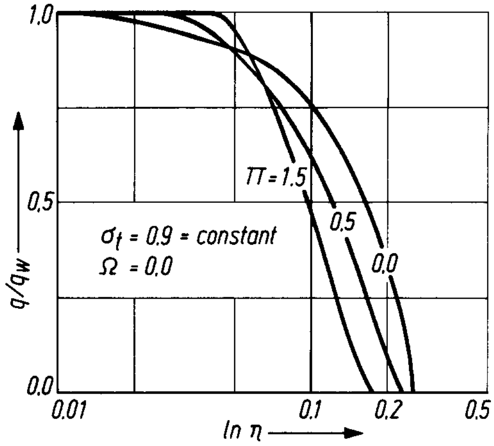

FIG. 9. Effect of the pressure gradient parameter (Π) on the heat flux distribution with constant turbulent Prandtl number (σ_t = 0·9) and zero wall heat flux gradient (Ω = 0).

profiles become flatter in the heat transfer case (Figs. 7 and 8). The influence of σ_t is, again, relatively insignificant. These curves were computed by assuming that the heat flux at the wall remained constant with x ($\Omega = 0$). By referring to equation (10) it can be seen that, in this case, $\xi = \Pi$. The distributions of the dimensionless heat flux ratio, q/q_w, for various values of Π and σ_t are illustrated in Figs. 9–11.

For the adiabatic wall, the variation of the

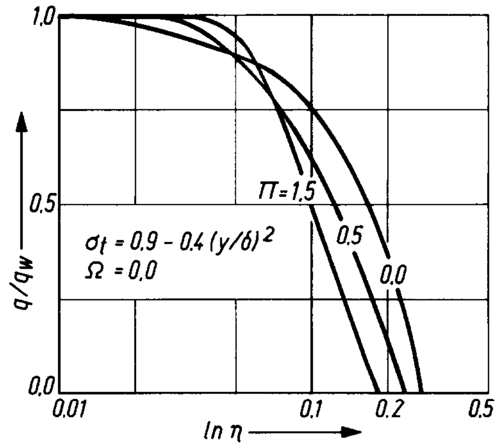

FIG. 10. Effect of the pressure gradient parameter (Π) and a variable turbulent Prandtl number (σ_t) on the heat flux distribution with zero wall heat flux gradient (Ω = 0).

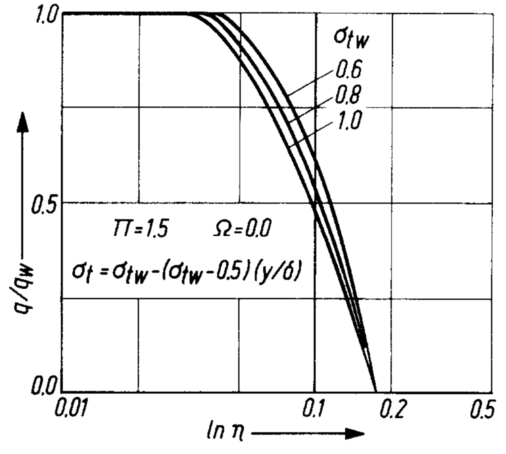

FIG. 11. Effect of the turbulent Prandtl number (σ_t) on the heat flux distribution with an adverse pressure gradient (Π = 1·5) and zero wall heat flux gradient (Ω = 0).

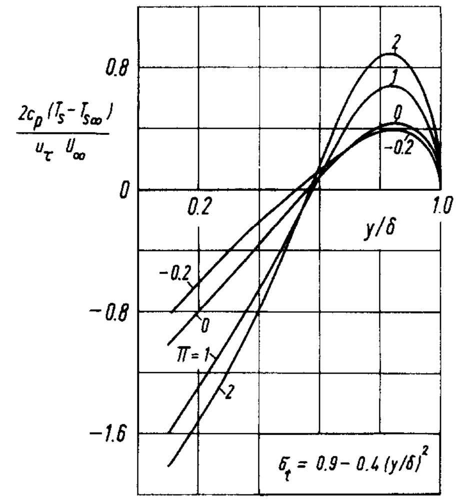

FIG. 12. Effect of the pressure gradient parameter on the total temperature distribution for the adiabatic wall case with variable turbulent Prandtl number.

total temperature through the boundary layer is investigated. By definition, the total temperature is

$$T_s = T + U^2/2c_p. \quad (40)$$

By making use of equations (10, (24) and (31) the above equation can be expressed in non-dimensional form as

$$\frac{2c_p(T_s - T_{s\infty})}{u_\tau U_\infty} = (\sigma_{tw} - 1)(2 - \omega F) F + \Psi_a. \quad (41)$$

The relative variation of the total temperature, according to the above equation, is shown in Figs. 12 and 13. In close proximity to the wall the total temperature must be smaller than the free stream total temperature ($T_s < T_{s\infty}$). The reason for this is that the adiabatic wall temperature is smaller than $T_{s\infty}$, and the local total temperature T_s must approach the adiabatic wall temperature as the wall is approached. However, in the outer layer the increase in turbulence causes a reversal in this trend, so that $T_s > T_{s\infty}$. These observations were previously noted and illustrated by Rotta [11]. The effect of an adverse pressure gradient is to increase the total temperature differences in both the inner and outer areas of the boundary layer (Fig. 12). It should be pointed out that the first term in equation (41) is influenced more by Π than the second term. This fact will be further explained in the discussion of the Recovery Factor in the next section. The effect of increasing or decreasing σ_{tw} is shown in Fig. 13.

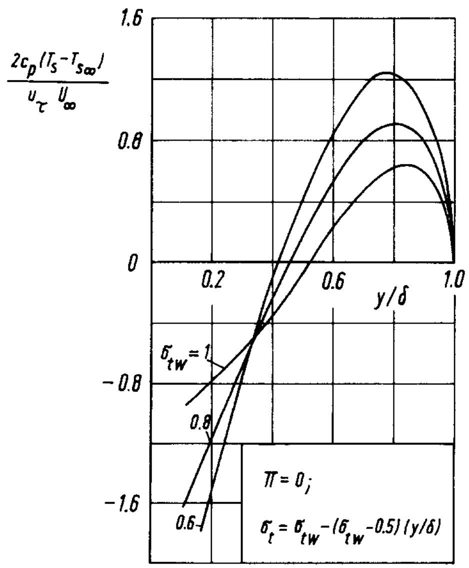

FIG. 13. Effect of the turbulent Prandtl number at the wall on the total temperature distribution with zero pressure gradient (Π = 0).

5.3 *Reynolds analogy and recovery factor*

With the help of the "friction temperature", defined by Squire, equation (5), the "temperature law of the wall" for heat transfer can be defined by

$$T = T_w - t_q \left[\sigma_{tw} \frac{U}{u_\tau} + (Pr - \sigma_{tw})\, a\right] \tag{42}$$

where a is a dimensionless quantity which, outside of the sublayer, is only a function of the ratio Pr/σ_{tw} and is constant [11]. The basic Reynolds analogy is a statement of equality between the rate of turbulent heat transfer and the rate of momentum transfer in the y direction at the wall. This equality only exists if $Pr = \sigma_{tw} = 1{\cdot}0$ and $\Pi = \Omega = 0$. The comparison is made through the use of the Stanton number.

$$St = \frac{-q_w}{\rho c_p U_\infty (T_w - T_\infty)} \tag{43}$$

and the friction coefficient

$$c_f/2 = \frac{\tau_w}{\rho U_\infty^2} \tag{44}$$

where, for $Pr = \sigma_t = 1{\cdot}0$ and $\Pi = \Omega = 0$, the Reynolds analogy factor, $2St/c_f$, is one. By using the temperature law of the wall, equation (42), the concept of the Reynolds analogy factor can be modified to include a more general class of flows. After substituting equation (42) into equation (43) the modified Reynolds analogy factor is

$$\frac{St}{c_f/2} = \frac{1}{\sigma_{tw} + \omega[\Psi_q + (Pr - \sigma_{tw})\, a]} \tag{45}$$

where Ψ_q is the asymptote of Ψ as the wall is approached. The variation of the Reynolds analogy factor with σ_t, Π and Ω is shown in Figs. 14–16. The manner in which $2\ St/c_f$ varies with Π and Ω is in general agreement with the computed results of Alber and Coats [12] and the experimental results of Bell [13].

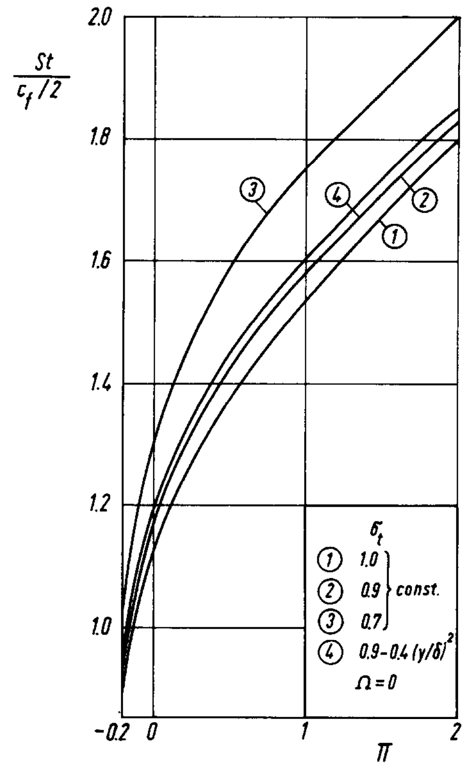

FIG. 14. Effect of the pressure gradient parameter (Π) on the Reynolds Analogy Factor for various turbulent Prandtl number distributions.

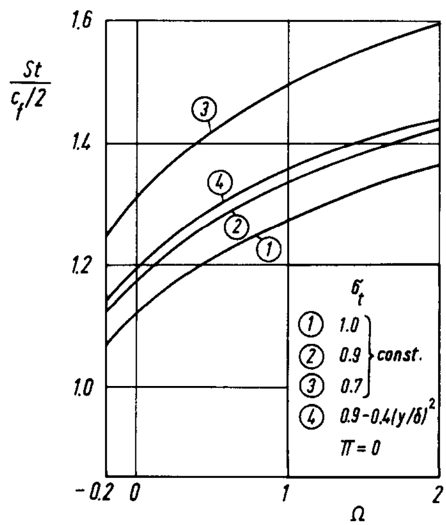

FIG. 15. Effect of the wall heat flux gradient parameter (Ω) on the Reynolds Analogy Factor for various turbulent Prandtl number distributions.

The increase of $2\,St/c_f$ with decreasing values of σ_{tw} (Fig. 16) was also noted in the computations made by Rotta [11] for $\Pi = \Omega = 0$.

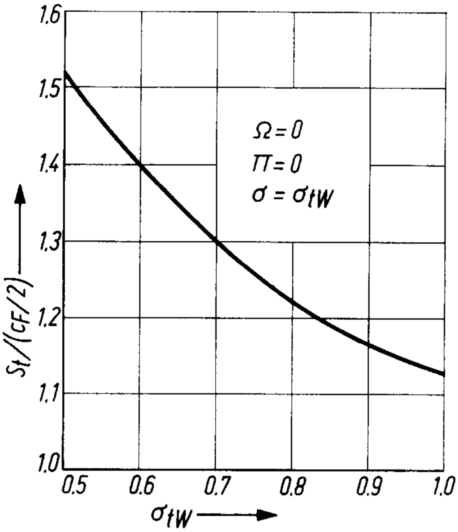

FIG. 16. Effect of the turbulent Prandtl number at the wall on the Reynolds Analogy Factor.

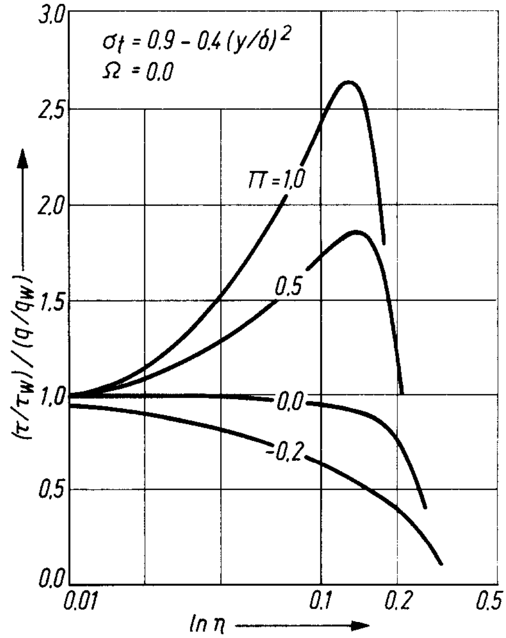

FIG. 17. Influence of the pressure gradient parameter (Π) on the shear stress–heat flux ratio with zero wall heat flux gradient ($\Omega = 0$).

The Reynolds analogy and the modified Reynolds analogy are basically comparisons of the shear stress and the heat flux at the wall. Of additional interest is the relative variation of these two quantities throughout the entire boundary layer. This comparison can be made in the outer part of the boundary layer through the use of the dimensionless ratio $(\tau/\tau_w)/(q/q_w)$, where τ/τ_w and q/q_w were previously defined by equations (20) and (39), respectively. This ratio can be expressed as

$$\frac{\tau/\tau_w}{q/q_w} = \frac{\sigma_t F'}{\Phi'}. \tag{46}$$

A plot of the ratio $(\tau/\tau_w)/(q/q_w)$, under the conditions considered in these sample calculations, rather clearly shows the relative influence of Π, Ω and σ_t (Figs. 17 and 18). In considering these profiles, it should be pointed out that a singularity exists at the outer edge of the boundary layer, where the dimensionless shear stress and heat flux ratios approach zero at the

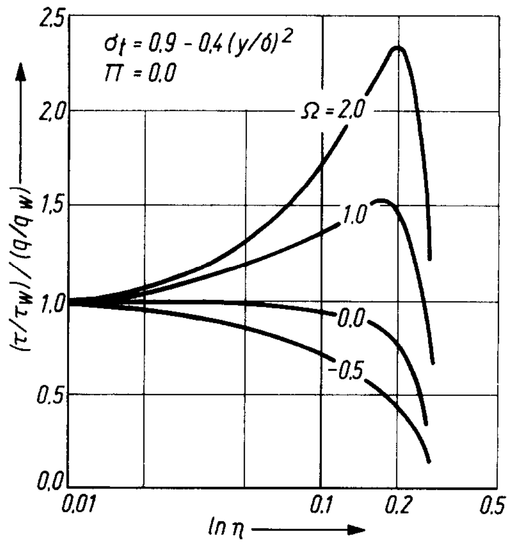

FIG. 18. Effect of the wall heat flux gradient parameter (Ω) on the shear stress–heat flux ratio with zero pressure gradient ($\Pi = 0$).

same point $\eta = \delta/\Delta$. This situation results from the fact that equation (46) was basically derived from equation (19), which leads to the assumption that the thermal and velocity boundary-layer thicknesses are equal. The effect of this assumption on the interpretation of the results cannot be precisely stated. However, the general effects of Π and Ω on the heat flux distributions are in agreement with the trends noted in the work of Alber and Coats [12], in which the assumption of equal thermal and velocity boundary layer thicknesses was not made. The profiles indicate the following general tendencies for incompressible turbulent boundary layers:

1. The turbulent shear stress in the outer part of the boundary layer increases much more than the turbulent heat flux when an adverse pressure gradient exists. This trend is reversed when $\Pi < 0$. Figure 17 illustrates these tendencies for constant σ_t.
2. With increasing values of wall heat flux gradient ($\Omega > 0$), the turbulent heat flux in the outer layer decreases (Fig. 18).
3. A comparison of Figs. 17 and 18 shows that wall heat flux and free stream pressure gradients have the same general effect on the heat flux distributions in the outer part of the boundary layer.

In a manner analogous to the heat-transfer case, a temperature "law of the wall" for the adiabatic wall case, can be given by

$$T = T_e - \sigma_{tw}\frac{U^2}{2c_p} - \frac{u_\tau^2}{2c_p}(Pr - \sigma_{tw})\, b \quad (47)$$

where T_e is the equilibrium temperature and b is analogous to a and is a dimensionless constant only dependent on the ratio Pr/σ_{tw} [11]. In the adiabatic wall case, the Recovery Factor, defined by

$$r = 2c_p(T_e - T_\infty)/U_\infty^2 \quad (48)$$

is a useful parameter to aid in the description of the temperature distribution through the boundary-layer. After substituting equations (10), (31) and (47) into equation (48) the Recovery Factor can be defined by

$$r = \sigma_{tw} + \omega\Psi_a + \omega^2(Pr - \sigma_{tw})\, b \quad (49)$$

where Ψ_a is the value of Ψ at the wall. As indicated in Fig. 19, the Recovery Factor does not appear to be affected by a free stream pressure gradient, within the range of values of Π which were used in the computations. At first glance, this does not appear to be consistent with the results illustrated in Fig. 12, where the pressure gradient parameter is seen to have a very significant effect on the total temperature distributions throughout the boundary layer. However, for a given σ_t and ω, the quantity plotted in Fig. 12 is a function of both F and Ψ (see equation (42)), while r is only a function of Ψ very close to the wall. As Π is varied from $-0{\cdot}2$ to $2{\cdot}0$, the magnitude of Ψ close to the wall

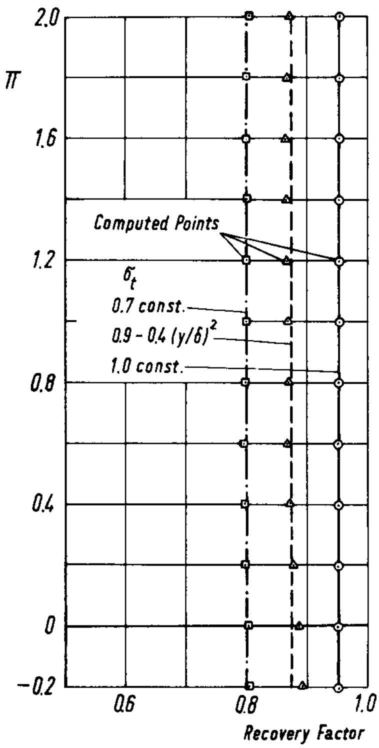

FIG. 19. Influence of the pressure gradient parameter (Π) and the turbulent Prandtl number at the wall (σ_{tw}) on the Recovery Factor.

increases very slightly for the assumed distributions for σ_t. Furthermore, when this small variation is multiplied by $\omega = 0{\cdot}04$, as in equation (49), the effect on r is barely discernible. In other words, the Recovery Factor does not appear to be sensitive to the so called "history effects" which were observed in the heat transfer case. However, the Recovery Factor is significantly influenced by the assumed value or distribution for σ_t (see Fig. 19). The distribution $\sigma_t = 0{\cdot}9 - 0{\cdot}4\,(y/\delta)^2$ yields a value of $r = 0{\cdot}88$, which agrees with previous theoretical and experimental results (e.g. Ref. [11]).

5.4 *Temperature distributions in expanding flows*

Measurements in compressible, expanding flows have indicated significant deviations from the Crocco theory [4], even in the absence of free stream pressure gradients. There has been much speculation about the basis or causes of these deviations. One theory is that the boundary layer at a given downstream point cannot be uncoupled from the upstream flow conditions. For example, if a pressure gradient exists upstream, it influences the conditions at downstream points even though $\Pi = 0$ at these downstream points. These effects are sometimes referred to as "history effects" and the speculation is that they influence the boundary layer in the same way that a pressure gradient through the whole flow field would [12]. Another possibility is that the turbulent Prandtl number cannot be assumed to be constant from the wall to free stream. In addition, even with the most careful and exact measuring techniques, it has not been possible to totally eliminate the effects of wall heat flux gradients.

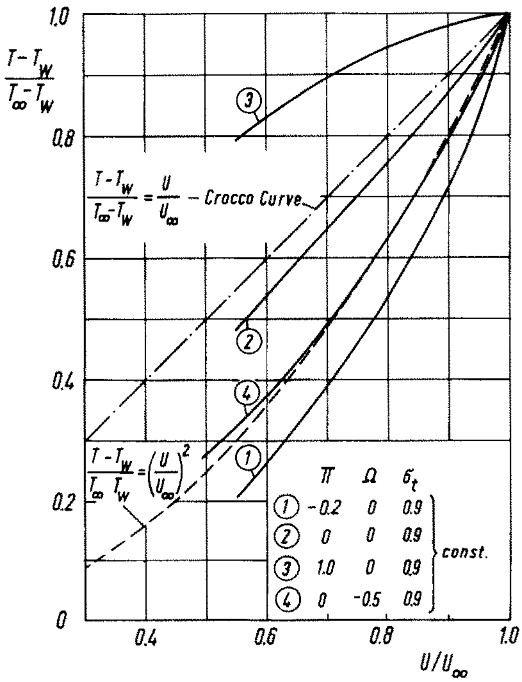

FIG. 20. Comparison of temperature distributions for expanding flows.

As already shown by the profiles previously presented (e.g. Figs. 2–4 etc.), the wall heat flux gradients have a very significant effect on the boundary layer temperature distributions and heat flux. Although the incompressible low speed solution for equilibrium boundary layers cannot be directly related to the supersonic case for expanding flows, the general tendencies of the flow to react to pressure gradients, wall heat flux gradients and turbulent Prandtl number would probably be the same. Therefore, these effects are shown in Fig. 20 for comparison with the Crocco theory. Since the considerations in this work are restricted to the outer turbulent part of the boundary layer, only about one-half of the profiles (solid lines) shown in Fig. 20, are the result of actual computations. Under the assumptions of a constant $\sigma_t = 0{\cdot}9$, a zero pressure gradient, $\Pi = 0$, and a negative wall heat flux gradient ($\Omega < 0$), the computations yield a curve which almost exactly corresponds to the quadratic curve in the outer part of the boundary layer. It can be argued that this agreement is strictly coincidental under the assumed conditions. However, previous experimental and theoretical research [9, 11] has indicated that the turbulent Prandtl number at the wall, σ_{tw}, is closer to 0·9 than 1·0. The profile for $\sigma_{tw} = 1{\cdot}0$ is shown for comparison in Fig. 20. The variation of σ_t is, again, of little significance in the outer layer. Furthermore, it would be reasonable to assume that, in most experimental investigations, a negative wall heat flux gradient exists (corresponding to decreasing wall heat flux with x). Therefore, if these two premises are accepted ($\sigma_t = 0{\cdot}9$ and $\Omega < 0$), the deviation of temperature profiles in turbulent boundary layers from the Crocco theory can be reasonably explained. In this regard, it should be noted that positive or negative free stream pressure gradients have the same general effect on the temperature distributions as do positive or negative wall heat flux gradients (Figure 20).

6. CONCLUSIONS

By applying the equilibrium or similarity concepts to incompressible turbulent boundary layers, the influence of free stream pressure gradients, wall heat flux gradients and varying turbulent Prandtl numbers on the boundary layer temperature profiles can be determined. Neglecting the viscous sublayer seems to have a more significant effect as the absolute value of the pressure gradients or wall temperature gradient increase. The turbulent heat flux decreases in comparison to the turbulent shear stress in the outer part of the boundary layer when positive wall heat flux or free stream pressure gradients exist. The general effects of the wall heat flux gradient parameter Ω and the pressure gradient parameter Π on the temperature distribution and heat flux in the boundary layer are the same. This is analogous to the condition in a laminar thermal boundary layer, where heating the wall has the same effect on boundary layer heat flux as a free stream pressure gradient [6]. The tendency to deviate from the Crocco theory, observed in compressible boundary layers, can be duplicated in the incompressible case by including the effects of negative free stream pressure or wall heat flux gradients and assuming that the turbulent Prandtl number $\sigma_t = 0{\cdot}9$. For the adiabatic wall case, the Recovery Factor does not appear to be influenced by adverse pressure gradients. However, it increases as the turbulent Prandtl number at the wall is increased.

Perhaps the most important conclusion to be made from these sample calculations is that the heat flux and temperature distributions in incompressible turbulent boundary layers are extremely dependent on the free stream pressure gradient and wall heat flux gradients. It can also be speculated that compressible turbulent boundary layers would be affected in much the same way.

The significance of this dependence on Π and Ω becomes evident when one considers that it is extremely difficult to design and carry out experimental investigations in which pressure gradients and wall heat flux gradients can be completely eliminated. Therefore, it would appear to be worthwhile to include these effects, under rigidly controlled conditions, in future experimental turbulent boundary layer measurements.

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DISTRIBUTION DE TEMPERATURE ET DE FLUX THERMIQUE DANS LES COUCHES LIMITES TURBULENTES EN EQUILIBRE ET INCOMPRESSIBLES

Résumé—Les concepts de couche limite en équilibre sont utilisés pour déterminer la distribution de température et les flux thermiques dans les couches limites turbulentes et incompressibles. On considère et discute les effets des gradients de flux thermique pariétaux, des gradients de pression dans l'écoulement libre et des nombres de Prandtl turbulents variables. Les deux cas du transfert thermique et de la paroi adiabatique sont analysés et on calcule le facteur d'analogie de Reynolds et le facteur thermique pariétal. Il est fait une comparaison entre la tension tangentielle et le flux thermique par turbulence. Une attention est portée à la zone externe turbulente entièrement développée de la couche limite.

TEMPERATUR- UND WÄRMESTROMVERTEILUNG IN INKOMPRESSIBLEN, TURBULENTEN GLEICHGEWICHTSGRENZSCHICHTEN

Zusammenfassung—Die Begriffe der Gleichgewichtsgrenzschichten werden verwendet, um Temperaturverteilungen und Wärmestrom in turbulenten, inkompressiblen Grenzschichten zu bestimmen. Die Effekte von Wandwärmestromgradienten, Freistromdruckgradienten und veränderlicher turbulenter Prandtl-Zahl werden betrachtet und diskutiert. Die zwei Fälle der Wärmeübertragung und der adiabaten Wand werden analysiert und der Reynolds-Analogiefaktor und der Rückgewinnfaktor werden berechnet. Es wird ein Vergleich gemacht zwischen der turbulenten Schubspannung und dem turbulenten Wärmestrom. Die Betrachtungen werden durch den äusseren, voll entwickelten Teil der Grenzschicht begrenzt.

РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ И ТЕПЛОВОГО ПОТОКА В НЕСЖИМАЕМЫХ ТУРБУЛЕНТНЫХ РАВНОВЕСНЫХ ПОГРАНИЧНЫХ СЛОЯХ

Аннотация—Понятия равновесных пограничных слоёв используются для определения распределений температуры и теплового потока в турбулентных несжимаемых пограничных слоях. Рассматривается и обсуждается эффект градиентов теплового потока на стенке, градиентов давления набегающего потока и изменения турбулентного числа Прандтля. Анализируются два случая переноса тепла и случай адиабатической стенки; рассчитываются на машине коэффициенты восстановления и аналогии Рейнольдса. Проведено сравнение между распределением сдвига и турбулентных тепловых потоков. Рассмотрения ограничены внешней полностью развитой турбулентной частью пограничных слоёв.